NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL NOTE 2457

AIR FORCES AND MOMENTS ON TRIANGULAR AND RELATED

WINGS WITH SUBSONIC LEADING EDGES OSCILLATING

IN SUPERSONIC POTENTIAL FLOW

By Charles E. Watkins

Langley Aeronautical Laboratory
Langley Field, Va.

NACA

Washington

September 1951

AQ M 00-10-3277

1

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2457

AIR FORCES AND MOMENTS ON TRIANGULAR AND RELATED

WINGS WITH SUBSONIC LEADING EDGES OSCILLATING

IN SUPERSONIC POTENTIAL FLOW

By Charles E. Watkins

SUMMARY

This analysis treats the air forces and moments in supersonic
potential flow on oscillating triangular wings and a series of swept-
back and arrow wings with subsonic leading edges and supersonic trailing
edges. For the wings undergoing sinusoidal torsional oscillations simul-
taneously with vertical translations, the linearized velocity potential
is derived in the form of a power series in terms of a frequency param-
eter. This method can be useful for treatment of similar problems for
other plan forms and for wings undergoing other sinusoidal motions.
For triangular wings, as many terms of such a series expansion as may
be desired can be determined; however, the terms after the first few
become very cumbersome.

Closed expressions that include the reduced frequency to the third
power, an order which is sufficient for a large class of practical
applications, are given for the velocity potential and for the components
of chordwise section force and moment coefficients.

These wings are found to exhibit the possibility of undamped tor-
sional oscillations for certain ranges of Mach number and locations of
the axis of rotation. The ranges of these parameters are delineated for
triangular wings.

INTRODUCTION

This paper is concerned with the derivation of expressions for the
velocity potential and associated forces and moments for oscillating
triangular wings in supersonic flow. The boundary-value problem for
the linearized velocity potential for an apex-forward triangular wing
oscillating in a supersonic main stream may be classified, according to
reference 1, as "purely supersonic" if the leading edges of the triangle

are outside the Mach cone emanating from the apex of the triangle or
"mixed supersonic" if the leading edges are inside this Mach cone.

In the purely supersonic case the principle of independence holds;
that is, the flow on the upper surface of the wing is independent of the
flow on the lower surface and vice versa. Garrick and Rubinow (refer-
ence 1) have shown that the boundary-value problem for the velocity
potential in the purely supersonic case can be satisfied by simple dis-
tributions of sources with local strength proportional to the local
prescribed normal velocity of the wing.

In the mixed supersonic case the principle of independence does not
hold. Boundary-value problems for lift-producing wings in this case can
be satisfied by distributions of doublets; the relation between doublet
strength and normal velocity of the wing is, however, in general, not
simple. The determination of this relation requires the solution of an
integral equation that employs the potential of a time-dependent unit
doublet as kernel and limits of integration that depend on Mach number
and wing plan form.

For treatment of problems that involve boundary conditions that are
independent of time, such as constant angle of attack, constant rate of
pitching, and so forth, the doublet potential, employed as kernel of the
integral equation, is considered independent of time and in these cases
the integral equations for triangular wings can be solved by a straight-
forward process.

For treatment of problems of oscillating wings, however, it is
necessary to employ, as the kernel of the integral equations, a doublet
potential that varies harmonically with time and in this case the solu-
tion of the integral equation, generally, becomes very cumbersome. If
the doublet potential or kernel is expanded in terms of the frequency
of oscillation, however, use can be made of knowledge of solutions of
integral equations for problems that are independent of time to obtain
an expanded form of solution for a wing undergoing harmonic oscillations.
Such a procedure was demonstrated in treatments of rectangular wings in
references 2 and 3. (The derivation in reference 3 is based on an
erroneous argument, regarding certain terms in the normal velocity;
nevertheless, the final expression given for the velocity potential is
correct.)

The purpose of the present paper is to make use of the expanded
form of the velocity potential to obtain the forces and moments, based
on the first few terms of this potential, for a rigid triangular wing
performing vertical and pitching sinusoidal oscillations in mixed super-
sonic flow. Although as many terms of the expanded potential as may be
desired can be obtained after the first few terms, the process becomes
very cumbersome. The flow normal to the leading edge is subsonic but

the flow normal to the trailing edge is considered to be supersonic. This latter consideration implies that the potential derived for triangular plan forms may be used to calculate the aerodynamic forces and moments for other plan forms that may be formed with the triangular wing by cutting the trailing edges so that they lie ahead of the Mach cones emanating from their foremost points.

Other approaches to the solution of the problem of oscillating triangular wings have been given by Robinson (reference 4), Haskind and Falkovich (reference 5), and by Stewartson (reference 6). In both references 4 and 5 formal solutions to the problem were obtained in terms of special systems of curvilinear coordinates. Robinson's solution was given in terms of a double summation of trilinear combinations of Bessel functions of the first kind with Lamé functions of the first and second kinds. Similarly, the solution of Haskind and Falkovich was given in terms of summations of Bessel functions of the first kind combined with elliptic integrals of the first and second kinds. In both references 4 and 5 the potentials were not reduced to useful forms for calculating forces and moments.

In reference 6 Stewartson makes an interesting though specialized use of the Laplace transformation to develop a method whereby terms of the velocity potential for triangular plan forms, expanded as herein, can be obtained. Stewartson gives formulas that, except for errors presumably in printing, can be used to develop the potential to the second power of the frequency but he omits many details in his derivation.

SYMBOLS

ϕ disturbance-velocity potential

x,y,z rectangular coordinates attached to wing moving in
        negative x-direction

ξ,η rectangular coordinates used to represent space location
        of doublets in xy-plane

Z_m function defining mean ordinates of any chordwise section
        of wing such as $y = y_1$ as shown in figure 1

$w(x,y_1,t)$ vertical velocity at surface of wing along chordwise
        section at $y = y_1$

x_0 abscissa of axis of rotation of wing as shown in figure 1

t	time
h	vertical displacement of axis of rotation
h_0	amplitude of vertical displacement of axis of rotation, positive downward
α	angle of attack
α_0	amplitude of angular displacement about axis of rotation, positive leading edge up
$\dot{h}, \dot{\alpha}$	time derivatives of h and α, respectively
V	velocity of main stream
c	velocity of sound
M	free-stream Mach number (V/c)
$\beta = \sqrt{M^2 - 1}$	
ω	frequency of oscillations
$\bar{\omega} = \dfrac{M^2 \omega}{V \beta^2}$	
k	reduced frequency $\left(\dfrac{b\omega}{V}\right)$
ϵ	half apex angle
$C = \tan \epsilon$	
a_{nm}	represents functions of $\bar{\omega}$, x, and M
D_n, F_n	functions used to denote doublet distribution functions
d_n	constants associated with D_n depending on βC
A_i, λ_j	constants depending on βC
σ_k	constants depending on βC and M
$2b$	root chord of wing

s   semispan of wing

σ slope of ray passing through vertex of wing

q, N, θ, τ dummy variables

ρ density

Δp local pressure difference

P   section force (total force at any spanwise station)

L_1, L_2, L_3, L_4 components of section force coefficients

M_α section moment (total moment about $x = x_0$ at any
                            spanwise station)

M_1, M_2, M_3, M_4 components of section moment coefficients

$\overline{M}_4$ total component of damping-moment coefficient

ANALYSIS

Boundary-Value Problem for the Velocity Potential

Referred to a rectangular coordinate system moving forward at a
uniform supersonic speed in the negative x-direction (see fig. 1) the
differential equation for the propagation of small disturbances that
must be satisfied by the velocity potential is

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{1}$$

The main governing boundary condition to be satisfied by the velocity
potential is that the flow be tangent to the surface of the wing, or

$$\left(\frac{\partial \phi}{\partial z}\right)_{z \to 0} = w(x,y,t) = V\frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \tag{2}$$

where Z_m is the vertical displacement of any point of the wing. For
the particular case of a wing independently performing small sinusoidal

pitching oscillations of amplitude α_0 about some spanwise axis x_0 and small sinusoidal vertical translations of amplitude h_0, the quantity Z_m in equation (2) is

$$Z_m = e^{i\omega t}\left[\alpha_0(x - x_0) + h_0\right] = \alpha(x - x_0) + h \tag{3}$$

(See fig. 1(b) for sketch showing instantaneous displacement of section $y = y_1$.) For convenience, the frequency of oscillation of both pitching and translatory motion is denoted by ω. Considering these motions to occur at separate frequencies would add no difficulties to the derivation.

Substituting the expression for Z_m (equation (3)) into equation (2) gives

$$w(x,y,t) = V\alpha + \dot{\alpha}(x - x_0) + \dot{h} \tag{4}$$

Equation (4) implies that the velocity potential may be expressed as the sum of separate effects due to position and motion of the wing associated with individual terms of this equation, namely

$$\phi = \phi_\alpha + \phi_{\dot{\alpha}} + \phi_{\dot{h}} \tag{5}$$

Derivation of ϕ

In order to obtain the analytical expression for the potential ϕ, it is necessary to derive only one of the subsidiary potentials appearing in equation (5), say ϕ_α. The other subsidiary potentials $\phi_{\dot{\alpha}}$ and $\phi_{\dot{h}}$ can then be obtained from the derived expression for ϕ_α by simple comparison.

In order to satisfy the boundary-value problem for ϕ_α, a convenient procedure is to start with the expanded form of the potential of a uniform distribution of doublets. Then, for a given power of the frequency of oscillation this potential, as will be shown in the following analysis, can be modified so that, when integration is made over the appropriate region, the results satisfy the differential equation (1) to the given power of the frequency and satisfy the condition of tangential flow exactly. The type of doublet required is that with its axis normal to the plane of the wing. The potential of such a doublet may be obtained

from the potential of a source, located in the plane of the wing, by partial differentiation with respect to the direction normal to the plane of the wing. Similarly, the potential of a distribution of the required type of doublets can be obtained from a distribution of sources located in the plane of the wing.

The potential at (x,y,z) due to sources located at points $(\xi,\eta,0)$ in region r (illustrated in fig. 2) of the xy-plane which satisfies the differential equation (1) may be written as

$$\phi_S = \frac{V\alpha}{\pi} \int_r \int \frac{e^{-i\overline{\omega}(x-\xi)} \cos\left(\frac{\overline{\omega}}{M} R\right)}{R} \, d\xi \, d\eta \qquad (6)$$

where

$$\overline{\omega} = \frac{M\omega}{C\beta^2} = \frac{M^2\omega}{V\beta^2}$$

and

$$R = \sqrt{(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2}$$

Expanding the integrand of equation (6) into a power series in $\overline{\omega}$, collecting terms with respect to ξ, and differentiating the resulting integral with respect to z gives the expanded form of the potential of a uniform distribution of doublets, namely

$$\phi_D = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \int_r \int \left[a_{01} \frac{1}{R} + a_{02}R + \ldots + a_{0m}R^{2m-3} + \ldots + \right.$$

$$\xi\left(a_{11}\frac{1}{R} + a_{12}R + \ldots + a_{1m}R^{2m-3} + \ldots\right) + \ldots +$$

$$\left. \xi^n\left(a_{n1}\frac{1}{R} + a_{n2}R + \ldots + a_{nm}R^{2m-3} + \ldots\right)\right] d\xi \, d\eta \qquad (7)$$

where

$$a_{n1} = \frac{(i\bar{\omega})^n}{n!} e^{i\bar{\omega}x}$$

(8)

and

$$a_{nm} = \frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2} a_{n1}$$

(9)

For convenience in the succeeding discussion and analysis equation (7) may be written in the following form:

$$\phi_D = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \sum_{n=0}^{\infty}\left[a_{n1} \int_r \int \frac{\xi^n}{R} \, d\xi \, d\eta \, + \right.$$

$$\left. \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2} a_{n1} \int_r \int \xi^n R^{2m-3} \, d\xi \, d\eta \right]$$

(10)

An interesting and significant property of equation (10) is that the coefficient of each power of ξ satisfies the differential equation (1) and has the form of a source potential with strength proportional to $\left(\frac{i\bar{\omega}}{n!}\right)^n e^{-i\bar{\omega}\xi}$. This property may be shown by writing the coefficient of ξ^n as follows:

$$\frac{a_{n1}}{R}\left[1 + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M} R\right)^{2m-2}\right] = \frac{a_{n1}}{R} \cos\left(\frac{\bar{\omega}}{M} R\right)$$

$$= \left(\frac{i\bar{\omega}}{n!}\right)^n e^{-i\bar{\omega}\xi}\left[\frac{e^{-i\bar{\omega}(x-\xi)} \cos\left(\frac{\bar{\omega}}{M} R\right)}{R}\right]$$

(11)

A more general solution to equation (1) may thus be obtained by introducing properly chosen weight or distribution functions (denoted by $\xi D_n(\xi,\eta)$) into the coefficients of ξ^n in equation (10). Let this solution be denoted by ϕ_1; then it can be written as

$$\phi_1 = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \sum_{n=0}^{\infty} \left[a_{nl} \int_r \int D_n(\xi,\eta) \frac{\xi^{n+1}}{R} \, d\xi \, d\eta + \right.$$

$$\left. \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{(2m-2)!} \left(\frac{\overline{\omega}}{M}\right)^{2m-2} a_{nl} \int_r \int D_n(\xi,\eta) \xi^{n+1} R^{2m-3} \, d\xi \, d\eta \right] \quad (12)$$

Examination of equation (12) shows that, at the surface $z = 0$, the potential ϕ_1 is determined by the first integral expression but that both integral expressions may give rise to normal velocity. In succeeding steps in this analysis it is shown that the distribution functions $D_n(\xi,\eta)$ in equation (12) can be determined so that the first integral expression taken alone will exactly satisfy the boundary condition of tangential flow for ϕ_α; that is $\left(\dfrac{\partial \phi_\alpha}{\partial z}\right)_{z=0} = V\alpha$. Also, any additional normal velocity that arises from the second integral expression can be canceled, to the required order, by consideration of additional doublet solutions to equation (1). The problem of satisfying the boundary-value problem for the velocity potential ϕ_α may thus be reduced to that of determining the appropriate distribution functions and additional solutions to equation (1).

In order to show that the first integral expression in equation (12) can be made to satisfy the boundary condition for ϕ_α, the coefficient $V\alpha a_{nl}$ appearing in this equation is first considered. If the analytical expression for the coefficient a_{nl} (equation (8)) is multiplied by x^n and summed with respect to n, the result is identically $V\alpha$. This result may be shown as follows:

$$V\alpha \sum_{n=0}^{\infty} x^n a_{nl} = V\alpha \sum_{n=0}^{\infty} \frac{(i\overline{\omega}x)^n}{n!} e^{-i\overline{\omega}x} = V\alpha e^{i\overline{\omega}x} e^{-i\overline{\omega}x} \equiv V\alpha \quad (13)$$

It is significant that this identity holds if only terms in $\overline{\omega}$ to any given power are considered. For example, retaining only terms including $\overline{\omega}$ to the third power gives

$$V\alpha\left[\left(1 - i\overline{\omega}x - \frac{\overline{\omega}^2 x^2}{2} + \frac{i\overline{\omega}^3 x^3}{6}\right) + x\left(i\overline{\omega} + \overline{\omega}^2 x - \frac{i\overline{\omega}^3 x^2}{2}\right) - x^2\left(\frac{\overline{\omega}^2}{2} - \frac{i\overline{\omega}^3}{2} x\right) - x^3 \frac{i\overline{\omega}^3}{6}\right] \equiv V\alpha \tag{14}$$

Next consider the normal velocity at $z = 0$ associated with ϕ_1, namely

$$w_1 = \left(\frac{\partial\phi_1}{\partial z}\right)_{z=0} = \frac{V\alpha}{\pi} \lim_{z \to 0} \sum_{n=0}^{\infty}\left[a_{n1} \int\int_r D_n(\xi,\eta) \frac{\xi^{n+1}}{R} d\xi\, d\eta + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\overline{\omega}}{M}\right)^{2m-2} a_{n1} \int\int_r D_n(\xi,\eta)\xi^{n+1} R^{2m-3} d\xi\, d\eta\right] \tag{15}$$

Examination of this equation and equation (13) reveals that the first integral expression on the right of equation (15) yields $V\alpha$ exactly, provided the distribution functions are determined so that the following integral equation is satisfied:

$$\frac{V\alpha}{\pi} \lim_{z \to 0} \frac{\partial^2}{\partial z^2} \int\int_r D_n(\xi,\eta) \frac{\xi^{n+1}}{R} d\xi\, d\eta = V\alpha x^n \tag{16}$$

The kernel of this integral equation has the form of a steady-state doublet potential. The problem of determining the distribution functions for this case is therefore analogous to determining distribution functions for certain steady-state problems. The distribution functions for steady-state problems, at least for those involving conical flow, can be determined by a straightforward process, the main details of which are given in the appendix. In this appendix a method of solving equation (16) for a triangular wing is derived and the distribution functions required to

derive the velocity potential for this wing to the third power of $\bar{\omega}$ are given. It is to be noted that the method derived for solving equation (16) for a triangular wing may be generalized to apply to various plan forms and to problems of satisfying the boundary conditions for various velocity distributions.

From this point on, the analysis is restricted to the derivation of terms of the expanded potential involving $\bar{\omega}$ to the third power. The method for deriving these first few terms is quite general and can be used to obtain as many additional terms of the expanded potential as may be desired. As previously pointed out, however, terms of the potential after the first few become very unwieldy.

If the appropriate distribution functions are known for terms involving $\bar{\omega}$ to the third power, equation (15) may be written as follows:

$$w_1 = V\alpha + w_2 \tag{17}$$

where

$$w_2 = -\frac{V\alpha}{\pi} \lim_{z \to 0} \frac{\partial^2}{\partial z^2} \frac{\overline{\omega}^2}{2M^2} \int_r \int \left[(1 - i\bar{\omega}x)\xi D_0 R + i\bar{\omega}\xi^2 D_1 R \right] d\xi \, d\eta \tag{18}$$

is the additional vertical velocity arising from the second integral expression in equation (12) involving $\bar{\omega}$ to the third power. In order to maintain the boundary condition for ϕ_α, this additional velocity w_2 must be canceled. As previously pointed out this canceling, to a required order, can be achieved by considering other doublet solutions to equation (1). For this particular case consider a relation $\bar{\phi}_1$ similar to ϕ_1 (equation (12)) having the following form:

$$\bar{\phi}_1 = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \frac{\overline{\omega}^2}{2M^2} \sum_{n=0}^{\infty} \left[a_{n1} \int_r \int \bar{D}_n(\xi,\eta)\xi^{n+3} \, d\xi \, d\eta + \right.$$

$$\left. \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{(2m-2)!} \left(\frac{\overline{\omega}}{M}\right)^{2m-2} a_{n1} \int_r \int \bar{D}_n(\xi,\eta)\xi^{n+3}R^{2m-3} \, d\xi \, d\eta \right] \tag{19}$$

The vertical velocity $\overline{w}_1$, to the third power of $\overline{\omega}$, arising from this
expression is

$$\overline{w}_1 = \frac{V\alpha}{2\pi} \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \left(\frac{\overline{\omega}}{M}\right)^2 \int_r \int \left[(1 - i\overline{\omega}x)\overline{D}_0 \frac{\xi^3}{R} + i\overline{\omega}\,\overline{D}_1 \frac{\xi^4}{R}\right] d\xi \; d\eta \quad (20)$$

In this equation the distribution functions $\overline{D}_0$ and $\overline{D}_1$ can be deter-
mined, as discussed subsequently, by the method given in the appendix so
that $\overline{w}_1$ is identically equal in value but opposite in sign to $\overline{w}_2$.
When these functions are determined, the boundary condition for ϕ_α is
satisfied by $w_1 + \overline{w}_1 = V\alpha$, which implies that the potential ϕ_α to the
third power of $\overline{\omega}$ is given by the sum

$$\phi_\alpha = \phi_1 + \overline{\phi}_1 \tag{21}$$

to this power of $\overline{\omega}$.

Expressions for $\overline{D}_0$ and $\overline{D}_1$ are given together with expressions
for other distribution functions D_0, D_1, D_2, and D_3 in the appendix.
In regard to the determination of $\overline{D}_0$ and $\overline{D}_1$, as well as other orders
for $\overline{D}_n$ in more extended treatments, it may appear necessary, in order
to formulate integral equations for these functions, to perform the
generally unwieldy integrations of the type

$$\lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \int_r \int D_n(\xi,\eta)\xi^{n+1}R^{2m-3} \; d\xi \; d\eta \tag{22}$$

appearing in equation (15). In general, however, the information neces-
sary for the determination of the functions $\overline{D}_n$ can be obtained, as is
done in the derivation of the functions D_n in the appendix, by exam-
ining the values of these integrals and their derivatives with regard to
the parameter $\theta = \frac{y}{x}$ at some particular value of θ.

Returning to equation (21) and introducing into this equation the
expression for ϕ_1 (equation (12)) and the expression for $\overline{\phi}_1$

(equation (19)), each to the third power of $\bar{\omega}$, gives for the potential ϕ_α to this power of $\bar{\omega}$ the following result:

$$\phi_\alpha = V\alpha \frac{\partial}{\partial z} \int_r \int \left\{ \xi D_0 \left[\left(1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2} x^2 + \frac{i\bar{\omega}^3}{6} x^3 \right) \frac{1}{R} - \left(\frac{\bar{\omega}^2}{2M^2} - \frac{i\bar{\omega}^3}{2M^2} x \right) R \right] + \right.$$

$$\xi^2 D_1 \left[\left(i\bar{\omega} + \bar{\omega}^2 x - \frac{i\bar{\omega}^3}{2} x^2 \right) \frac{1}{R} - \frac{i\bar{\omega}^3}{2M^2} R \right] - \xi^3 D_2 \left(\frac{\bar{\omega}^2}{2} - \frac{i\bar{\omega}^3}{3} x \right) \frac{1}{R} -$$

$$\left. \xi^4 D_3 \frac{i\bar{\omega}^3}{6} \frac{1}{R} + \xi^3 \bar{D}_0 \left(\frac{\bar{\omega}^2}{2M^2} - \frac{i\bar{\omega}^3}{2M^2} x \right) \frac{1}{R} + \xi^4 \bar{D}_1 \frac{i\bar{\omega}^3}{2M^2} \frac{1}{R} \right\} d\xi \, d\eta \qquad (23)$$

Since it may be shown, as in reference 7, for example, that

$$\frac{V\alpha}{\pi} \lim_{z \to 0} \frac{\partial}{\partial z} \int_r \int D_n \xi^{n+1} \, d\xi \, d\eta = V\alpha x^n D_n(x,y)$$

equation (23) reduces, at $z = 0$, to

$$\phi_\alpha = V\alpha \left[\left(1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2} x^2 + \frac{i\bar{\omega}^3}{6} x^3 \right) D_0(x,y) + \right.$$

$$x \left(i\bar{\omega} + \bar{\omega}^2 x - \frac{i\bar{\omega}^3 x^2}{2} \right) D_1(x,y) - x^2 \left(\frac{\bar{\omega}^2}{2} - \frac{i\bar{\omega}^3}{3} x \right) D_2(x,y) -$$

$$\left. x^3 \frac{i\bar{\omega}^3}{6} D_3(x,y) + x^2 \left(\frac{\bar{\omega}^2}{2M^2} - \frac{i\bar{\omega}^3}{2M^2} x \right) \bar{D}_0(x,y) + x^3 \frac{i\bar{\omega}^3}{2M^2} \bar{D}_1(x,y) \right] \qquad (24)$$

which, after the expressions for the distribution functions given in the
appendix are substituted and the terms are regrouped, may be written in
the following simple form:

$$\phi_\alpha = V\alpha\sqrt{c^2x^2 - y^2}\left[A_0 + (A_1 - A_0)i\bar{\omega}x - \sigma_1\bar{\omega}^2x^2 - \sigma_2\bar{\omega}^2\beta^2y^2 + \right.$$

$$\left. \sigma_3 i\bar{\omega}^3x^3 + \sigma_4 i\bar{\omega}^3\beta^2y^2x\right] \tag{25}$$

The quantities A_0, A_1, and σ_1, σ_2, σ_3, and σ_4 in this equation
are part of a group of quantities A_i, λ_j, σ_k, which are defined in
the appendix; these quantities are suitable for writing the potential
and expressions subsequently derived for forces and moments in simple form.
The quantities A_i and λ_j are functions only of the product βC (ratio
of tangent of the half apex angle of the triangle to the Mach angle) and
are shown plotted in figures 3 and 4, respectively. The quantities σ_k
are functions of Mach number M and the product βC. These quantities
may be evaluated for particular values of M from the plots of λ_j in
figure 4.

The quantities A_0 and A_1 are the same, as should be expected,
as the parameters associated, respectively, with constant angle of attack
and constant pitching of triangular wings of references 8 and 9.

Expressions for the potentials $\phi_{\dot{\alpha}}$ and $\phi_{\dot{h}}$ can be obtained by the
method discussed for obtaining ϕ_α, or they can be obtained to the third
order of $\bar{\omega}$ by comparison and synthesis from equation (23). After
simplification these expressions are

$$\phi_{\dot{\alpha}} = \dot{\alpha}\left[\sqrt{c^2x^2 - y^2}\left(A_1x - 2\lambda_2 i\bar{\omega}x^2 + A_3 i\bar{\omega}\beta^2y^2 - \sigma_5\bar{\omega}^2x^3 + \sigma_6\bar{\omega}^2\beta^2y^2x\right) - \right.$$

$$\left. x_0\sqrt{c^2x^2 - y^2}\left(A_0 - 2\lambda_1 i\bar{\omega}x - \sigma_1\bar{\omega}^2x^2 - \sigma_2\bar{\omega}^2\beta^2y^2\right)\right] \tag{26}$$

$$\phi_{\dot{h}} = \dot{h}\left[\sqrt{c^2x^2 - y^2}\left(A_0 - 2\lambda_1 i\bar{\omega}x - \sigma_1\bar{\omega}^2x^2 - \sigma_2\bar{\omega}^2\beta^2y^2\right)\right] \tag{27}$$

At $\beta C = 1$ or $C = \frac{1}{\beta}$, which is the condition at which the Mach lines from the apex of the triangle coincide with the leading edges of the triangle, equations (23), (26), and (27) reduce, respectively, to

$$\phi_\alpha = \frac{2V\alpha}{\beta\pi}\sqrt{x^2 - \beta^2 y^2}\left(1 - \frac{i\bar\omega x}{3} - \frac{7M^2 + 5}{90M^2}\bar\omega^2 x^2 - \frac{2M^2 - 5}{90M^2}\bar\omega^2\beta^2 y^2 + \right.$$

$$\left. \frac{3M^2 + 7}{210M^2}i\bar\omega^3 x^3 + \frac{2M^2 - 7}{210M^2}i\bar\omega^3\beta^2 y^2 x\right) \tag{28}$$

$$\phi_{\dot\alpha} = \frac{2\dot\alpha}{\beta\pi}\sqrt{x^2 - \beta^2 y^2}\left[\left(\frac{2}{3}x - \frac{8i\bar\omega}{45}x^2 + \frac{2i\bar\omega}{45}\beta^2 y^2 - \frac{\bar\omega^2(11M^2 + 7)}{315M^2}x^3 + \right.\right.$$

$$\left.\left. \frac{\bar\omega^2(2M^2 + 7)}{315M^2}\beta^2 y^2 x\right) - x_0\left(1 - \frac{i\bar\omega x}{3} - \frac{7M^2 + 5}{90M^2}\bar\omega^2 x^2 - \frac{2M^2 - 5}{90M^2}\bar\omega^2\beta^2 y^2\right)\right] \tag{29}$$

$$\phi_{\dot h} = \frac{2\dot h}{\beta\pi}\sqrt{x^2 - \beta^2 y^2}\left(1 - \frac{i\bar\omega x}{3} - \frac{7M^2 + 5}{90M^2}\bar\omega^2 x^2 - \frac{2M^2 - 5}{90M^2}\bar\omega^2\beta^2 y^2\right) \tag{30}$$

For $M = 1$ or for values of the product βC such that $\beta^2 C^2 \ll 1$, equations (23), (26), and (27) reduce, respectively, to

$$\phi_\alpha = V\alpha\sqrt{C^2 x^2 - y^2} \tag{31}$$

$$\phi_{\dot\alpha} = \dot\alpha\sqrt{C^2 x^2 - y^2}\left(x - x_0\right) \tag{32}$$

$$\phi_{\dot{h}} = \dot{h}\sqrt{c^2 x^2 - y^2} \qquad (33)$$

These expressions are the counterpart of the potential for vanishingly small triangular wings in steady flow (reference 10) and may thus be regarded as the potential of a small triangle oscillating in pitch and vertical translation in either subsonic or supersonic flow.

Forces and Moments

As pointed out in the introduction, the velocity potential for the triangular wing can be used to calculate the aerodynamic forces and moments for other plan forms that can be formed from the triangular wing by cutting the trailing edges so that they lie ahead of the Mach cones emanating from their foremost points. Sketches of different plan forms thus obtained are shown in figure 5.

The force and moment coefficients desirable for most flutter calculations are those that yield the spanwise variation in these quantities or chordwise force and moment coefficients. These coefficients are obtained by integrating the pressure difference along any chord for the forces and the pressure difference multiplied by a moment arm for the moments. A convenient procedure in deriving these quantities is to introduce the reduced-frequency parameter $\dfrac{b\omega}{V} = k$ and to employ the the variables x, y, and x_0 in a new sense as nondimensional quantities obtained by dividing the old variables by the maximum chord $2b$ of the wing.

The pressure difference between the upper and lower surfaces of the wing is

$$\Delta p = -2\rho\left(\frac{V}{2b}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right) \qquad (34)$$

The force, positive downward, at any section of any of the plan forms shown in figure 5 may be expressed as

$$P = 2b \int_{y/C}^{x_1} \Delta p \ dx = -4\rho b \int_{y/C}^{x_1} \left(\frac{V}{2b} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \right) dx$$

$$= -2\rho V \left\{ \phi \Big]_{y/C}^{x_1} + 2ik \int_{y/C}^{x_1} \phi \ dx \right\} \tag{35}$$

where x_1 has the following values with respect to the different configurations shown in figure 5:

In plan form A

$$x_1 = 1 \tag{36a}$$

In plan form B

$$x_1 = 1 - \frac{y}{m} \tag{36b}$$

In plan form C

$$x_1 = 1 + \frac{y}{m} \tag{36c}$$

In plan form D

$$\left. \begin{aligned} x_1 &= 1 + \frac{y}{m} \quad \text{for} \quad 0 \leq y \leq m\left(\frac{s}{2bC} - 1 \right) \\ x_1 &= \frac{s}{2bC} \quad \text{for} \quad m\left(\frac{s}{2bC} - 1 \right) < y \leq \frac{s}{2b} \end{aligned} \right\} \tag{36d}$$

After the expression for ϕ, given by the sum of equations (25), (26), and (27), is substituted into equation (35) and the integration is performed, the results may be reduced to the form

$$P = -4\rho b V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (L_1 + iL_2) + \alpha_0 (L_3 + iL_4) \right] \tag{37}$$

where

$$L_1 = \frac{x_1 \sqrt{C^2 x_1^2 - y^2}}{\beta^2} \left(4M^2 \lambda_1 - \beta^2 A_0 \right) + \frac{A_0 y^2}{C} \cosh^{-1} \frac{C x_1}{y} \tag{38a}$$

$$L_2 = \sqrt{c^2 x_1^2 - y^2}\left[\frac{A_0}{k} - \frac{4M^2 k x_1^2}{3\beta^4}\left(3M^2\sigma_1 - 2\beta^2\lambda_1\right) - \frac{4M^2 k y^2}{3\beta^2 c^2}\left(3M^2 c^2\sigma_2 + 2\beta^2\lambda_1\right)\right]$$

$$(38b)$$

$$L_3 = \sqrt{c^2 x_1^2 - y^2}\left\{\frac{A_0}{k^2} + \frac{4x_1^2}{3\beta^4}\left(6\beta^2 M^2\lambda_2 - \beta^4 A_1 - 3M^4\sigma_1 + 2M^2\beta^2\lambda_1\right) + \right.$$

$$\left. \frac{4y^2}{3\beta^2 c^2}\left[\beta^2 A_1 - 3\beta^2 M^2 c^2 A_3 - 3M^4 c^2\sigma_2 - 2M^2\lambda_1\right]\right\} - 2x_0 L_1 \qquad (38c)$$

$$L_4 = \sqrt{c^2 x_1^2 - y^2}\left[\frac{x_1}{\beta^2 k}\left(2\beta^2 A_1 + \beta^2 A_0 - 4M^2\lambda_1\right) + \frac{2M^2 k x_1^3}{\beta^6}\left(2\beta^4\lambda_2 - \right.\right.$$

$$M^2\beta^2\sigma_1 + 4M^4\sigma_3 - 4\beta^2 M^2\sigma_5\right) - \frac{M^2 k x_1 y^2}{\beta^4 c^2}\left(4\beta^4 c^2 A_3 + 2\beta^2\lambda_2 - M^2\sigma_1 + \right.$$

$$4M^2\beta^2 c^2\sigma_2 - 8M^4 c^2\sigma_4 - 8\beta^2 c^2 M^2\sigma_6\right)\right] - \cosh^{-1}\frac{c x_1}{y}\left[\frac{A_0 y^2}{kC} - \right.$$

$$\left.\frac{M^2 k}{\beta^4 c^3} y^4\left(4\beta^2 c^2 A_3 - 2\beta^2\lambda_2 + M^2\sigma_1 + 4M^2\beta^2 c^2\sigma_2\right)\right] - 2x_0 L_2 \qquad (38d)$$

In a similar manner, the moment (positive leading edge up) about the axis $x = x_0$ is

$$M_\alpha = -4b^2 \int_{y/C}^{x_1} (x - x_0)\,\Delta p\,dx$$

$$= -4\rho V^2 k^2 b^2\left[\frac{h_0}{b}(M_1 - iM_2) + \alpha_0(M_3 + iM_4)\right] \qquad (39)$$

where

$$M_1 = \sqrt{C^2 x_1{}^2 - y^2} \left\{ \frac{4x_1{}^2}{3\beta^2}\left(4M^2\lambda_1 - \beta^2 A_0\right) + \frac{4y^2}{3\beta^2 C^2}\left(2M^2\lambda_1 + \beta^2 A_0\right) \right\} - 2x_0 L_1 \quad (40a)$$

$$M_2 = \sqrt{C^2 x_1{}^2 - y^2}\left[\frac{A_0}{k}x_1 + \frac{2M^2 k x_1{}^3}{\beta^4}\left(2\beta^2\lambda_1 - 3M^2\sigma_1\right) - \right.$$

$$\left. \frac{M^2 k x_1{}^2}{\beta^4 C}\left(2\beta^2\lambda_1 + M^2\sigma_1 + 4M^2\beta^2 C^2\sigma_2\right)\right] + \left[\frac{A_0 y^2}{Ck} - \right.$$

$$\left. \frac{M^2 k}{\beta^4 C^3}y^4\left(2\beta^2\lambda_1 + M^2\sigma_1 + 4M^2 C^2\sigma_1\right)\right]\cosh^{-1}\frac{Cx_1}{y} - 2x_0 L_2 \quad (40b)$$

$$M_3 = \sqrt{C^2 x_1{}^2 - y^2}\left[\frac{A_0 x_1}{k^2} - \frac{2x_1{}^3}{\beta^4}\left(\beta^4 A_1 - 2M^2\beta^2\lambda_1 - 6M^2\beta^2\lambda_2 + 3M^4\sigma_1\right) + \right.$$

$$\left. \frac{x_1 y^2}{\beta^4 C^2}\left(\beta^4 A_1 - 4M^2\beta^4 C^2 A_3 - 2\beta^2 M^2\lambda_1 + 2M^2\beta^2\lambda_2 - M^4\sigma_1 - 4M^4\beta^2 C^2\sigma_2\right)\right] +$$

$$\left[\frac{A_0 y^2}{Ck^2} + \frac{y^4}{\beta^4 C^3}\left(\beta^4 A_1 - 4M^2\beta^4 C^2 A_3 - 2M^2\beta^2\lambda_1 + 2M^2\beta^2\lambda_2 - M^4\sigma_1 - \right.\right.$$

$$\left.\left. 4M^4\beta^2 C^2\sigma_2\right)\right]\cosh^{-1}\frac{Cx_1}{y} - 2x_0\left(M_1 + L_3 + 2x_0 L_1\right) \quad (40c)$$

$$M_4 = \sqrt{c^2 x_1^2 - y} \left[\frac{4x_1^2}{3\beta^2 k} \left(\beta^2 A_0 + 2\beta^2 A_1 - 4M^2\lambda_1 \right) - \frac{4y^2}{3\beta^2 c^2 k} \left(\beta^2 A_0 - \beta^2 A_1 + \right. \right.$$

$$\left. 2M^2\lambda_1 \right) + \frac{16M^2 k x_1^4}{5\beta^6} \left(2\beta^4\lambda_2 - \beta^2 M^2\sigma_1 + 4M^4\sigma_3 - 4M^2\beta^2\sigma_5 \right) -$$

$$\frac{16M^2 k x_1^2 y^2}{15\beta^6 c^2} \left(5\beta^6 c^2 A_3 + 2\beta^4\lambda_2 - M^2\beta^2\sigma_1 + 5M^2\beta^4 c^2\sigma_2 - M^4\sigma_3 - \right.$$

$$10M^4\beta^2 c^2\sigma_4 + M^2\beta^2\sigma_5 - 10M^2\beta^4 c^2\sigma_6 \left. \right) + \frac{16M^2 k y^4}{15\beta^6 c^2} \left(5\beta^6 c^2 A_3 - \right.$$

$$4\beta^4\lambda_2 + 2M^2\beta^2\sigma_1 + 5M^2\beta^4 c^2\sigma_2 + 2M^4\sigma_3 + 5M^4\beta^2 c^2\sigma_4 -$$

$$\left. \left. 2M^2\beta^2\sigma_5 + 5M^2\beta^4 c^2\sigma_6 \right) \right] - 2x_0 \left(M_2 + L_4 + 2x_0 L_2 \right) \qquad (40d)$$

In equation (37), for example, the quantity $(L_1 + iL_2)$ is the lift-force coefficient associated with vertical motion of the wing. The real part L_1 is in phase with the vertical position of the wing and the complex part L_2 is $90°$ out of phase with this position. Similar definitions apply to the lift coefficient $(L_3 + iL_4)$ associated with pitching motion and to the moment coefficients $(M_1 + iM_2)$ and $(M_3 + iM_4)$. The complex or out-of-phase terms determine the aerodynamic damping associated with different wing motions.

Although the expressions for the components of lift-force and moment coefficients in equations (38) and (40), respectively, are lengthy, they may be quite easily evaluated with the aid of the graphs in figures 3 and 4.

DISCUSSION

Sample Calculations

In order to give some indication as to the general nature of the spanwise distribution of the different components of lift and moment coefficients, equations (38) and (40) have been evaluated at different spanwise positions y for plan form A (fig. 5(a)) for the following set of conditions: $\beta C = 0.5$, $C = 1.0$, $x_0 = 0.6$, $M = \sqrt{5/4}$, and $k = 0.1$. These sample results are plotted as functions of spanwise position in figure 6. The spanwise variations of the different components of lift force are shown in figure 6(a) and the corresponding variations of moment coefficients in figure 6(b).

In figure 6 note that, for the particular set of conditions for this example, the maximum values of the components of moment coefficients M_1, M_2, and M_3 are positive and act near the tips of the wing, whereas the maximum value of the component M_4 is negative but also acts near the tips. It may also be noted that the integrated (in spanwise direction) values of the components of moment coefficient, or components of total moment coefficient, would in each case have the same sign as the maximum value of the corresponding component of section moment coefficient. This result is not necessarily true in general, because changing some of the parameters involved in the evaluation of the spanwise distribution of some components of both force and moment coefficients may change the distributions significantly from those shown in figure 6.

The fact that the total component of moment coefficient M_4 is negative in the example just discussed shows that, for the conditions of the example, this term would not contribute to the aerodynamic damping but, on the contrary, would act as a source of energy for the oscillating system. This circumstance is significant since it leads to the possibility of the single-degree-of-freedom torsional instability discussed in the following paragraphs.

Undamped Torsional Oscillations

The wing plan forms discussed herein, like two-dimensional and rectangular wings, exhibit the possibility of undamped torsional oscillations for certain ranges of Mach number M and location of axis of rotation x_0. This fact is borne out, as indicated in the preceding paragraph, by considering the integrated (spanwise) value of the component of damping moment M_4 associated with pitching or torsional

motions. The main results of this phenomenon can be obtained by considering very slow oscillations so that only terms in equation (40d) for M_4 involving the reduced frequency k to the order $1/k$ need be retained. In this case,

$$M_4 = \frac{\sqrt{Cx_1^2 - y^2}}{k}\left\{-\frac{4x_1^2}{\beta^2}\left[(M^2 + 1)A_0 - A_1(4M^2 - 2)\right] + \frac{8(2M^2 - 1)}{3\beta^2 C^2}\lambda_1 y^2 + \right.$$

$$\left. \frac{4x_0}{\beta^2}\left[\left(A_0 - (2M^2 - 1)A_1\right)x_1 + \beta^2 x_0 A_0\right]\right\} \tag{41}$$

For the triangular plan form (fig. 5(a)) the integrated value or total component of moment coefficient is

$$\overline{M}_4 = \frac{\pi C^2}{2\beta^2 k}\left\{-(2M^2 + 1)A_0 + 3(2M^2 - 1)A_1 + 4x_0\left[A_0 - (2M^2 - 1)A_1\right] + \right.$$

$$\left. 4x_0^2(M^2 - 1)A_0\right\} \tag{42}$$

In general, the condition of torsional stability or instability depends on the sign and magnitude of $\overline{M}_4$ in equation (42). Positive values of $\overline{M}_4$ indicate stable conditions and negative values indicate the possibility of torsional instability. Between the stable and unstable conditions, that is when $\overline{M}_4$ vanishes, a borderline state of unstable equilibrium separating damped and undamped torsional oscillations exists.

The ranges of values of Mach number M and location of axis of rotation x_0 for which $\overline{M}_4$ vanishes for some selected values of $C = \tan \epsilon$ are shown plotted in figure 7. The regions inside the curve in this figure indicate instability. The dashed curve, on which some of the solid curves terminate, represents the locus of values of M and x_0 for which $\overline{M}_4$ vanishes for the whole class of triangular wings with supersonic leading edges, that is, for triangular wings where $\beta C \geq 1$. It will be noted that $\overline{M}_4$ vanishes for values of x_0 ahead of the root $\frac{3}{4}$-chord position. It will also be noted that, as the vertex

angle $\epsilon = \tan^{-1} C$ decreases to 30°, the range of values of Mach number for which $\overline{M}_4$ vanishes decreases sharply.

In conclusion, investigation of equation (42) shows that, for a given value of the reduced frequency k, Mach number M, and location of the axis of rotation x_0, the magnitude of the damping coefficient $\overline{M}_4$ decreases as C decreases and, consequently, torsional instability is less likely to occur with slender triangles than with wider triangles.

Langley Aeronautical Laboratory
    National Advisory Committee for Aeronautics
        Langley Field, Va., June 19, 1951

APPENDIX

METHOD FOR DETERMINING DISTRIBUTION FUNCTIONS

In this appendix a method of obtaining the distribution functions is developed in detail. Expressions for the distribution functions required to derive the velocity potential to the third power of the frequency of oscillations and a set of functions useful in writing the expression for the potential in concise form are given.

As indicated in references 8 and 9, where the triangular wing is treated for constant angle of attack and for constant rolling and pitching motions, a convenient form of the velocity potential corresponding to a distribution of vertical velocity proportional to x^n is

$$\phi_n = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{-C}^{C} F_n(\sigma) \, d\sigma \int_0^{\xi_1} \frac{\xi^{n+2} \, d\xi}{\sqrt{(x - \xi)^2 - \beta^2(y - \sigma\xi)^2 - \beta^2 z^2}} \qquad (A1)$$

where σ is the slope of a ray passing through the vertex of the wing $(\eta = \sigma\xi)$, ξ_1 is the least value of ξ that causes the denominator R in the integrand to vanish, and $F_n(\sigma) = \frac{1}{\xi} D_n(\xi, \eta)$ is the distribution function that is to be determined so that

$$(w_n)_{z=0} = \left(\frac{\partial \phi_n}{\partial z}\right)_{z=0} = x^n \qquad (A2)$$

In equation (A1), the integrand is noted to be singular at the limit $\xi = \xi_1$. A form of the integrand which avoids this difficulty is obtained by making the following change of variables:

$$\left. \begin{aligned}
q &= \frac{x}{1 - \beta^2\sigma^2} (1 - \beta^2\sigma\theta) \\[2ex]
N &= \frac{\beta x}{1 - \beta^2\sigma^2} \sqrt{(\theta - \sigma)^2 + \frac{z^2}{x^2}(1 - \beta^2\sigma^2)} \\[2ex]
\theta &= \frac{y}{x} \\[2ex]
\tau &= \cosh^{-1} \frac{q - \xi}{N}
\end{aligned} \right\} \qquad (A3)$$

where τ is the new variable of integration. With these substitutions equation (A1) becomes

$$\phi_n = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{-C}^{C} \frac{F_n(\sigma)}{\sqrt{1 - \beta^2 \sigma^2}} \int_0^{\cosh^{-1}\frac{q}{N}} (q - N \cosh \tau)^{n+2} \, d\tau \tag{A4}$$

and the corresponding expression for $w_n = \left(\dfrac{\partial \phi_n}{\partial z}\right)_{z=0}$, which is an integral equation for $F_n(\sigma)$, is

$$w_n = \frac{1}{\pi} \lim_{z \longrightarrow 0} \left\{ -\beta^2(n+2) \int_{-C}^{C} \frac{F_n(\sigma)}{\left(1 - \beta^2\sigma^2\right)^{3/2}} \int_0^{\cosh^{-1}\frac{q}{N}} N^n\left(\frac{q}{N} - \cosh\tau\right)^{n+1} \cosh\tau \, d\tau \; + \right.$$

$$\beta^4 z^2(n+2) \int_{-C}^{C} \frac{F_n(\sigma)\, d\sigma}{\left(1 - \beta^2\sigma^2\right)^{5/2}} \int_0^{\cosh^{-1}\frac{q}{N}} N^{n-2}\left(\frac{q}{N} - \cosh\tau\right)^{n}\left(\frac{q}{N} + n\cosh\tau\right) \cosh\tau \, d\tau$$

$$= x^n \tag{A5}$$

In this equation the value of w_n obtained by performing the indicated integrations and then going to the limit $z = 0$ is the same as the value that would be obtained by first going to the limit and then performing the integration but neglecting singularities, pointed out subsequently, that arise when the value of σ approaches the value of θ. Making use of this fact reduces the calculations involved and simplifies the integral equation for F_n. Thus passing to the limit gives for equation (A5)

$$w_n = -\frac{\beta^2(n+2)}{\pi} \int_{-C}^{C} \frac{F_n(\sigma)\,d\sigma}{\left(1-\beta^2\sigma^2\right)^{3/2}} \int_0^{\cosh^{-1}\frac{q}{N_0}} N_0{}^n \left(\frac{q}{N_0} - \right.$$

$$\left. \cosh\tau\right)^{n+1} \cosh\tau\,d\tau$$

$$= x^n \tag{A6}$$

where N_0 is the value of N at $z = 0$. The requirement that the normal velocity be proportional to x^n and independent of y implies that all derivatives of equation (A6) with respect to y or θ must vanish, or that $F_n(\sigma)$ must be so determined that the final value of the integral in this equation be independent of θ. The requirement that all derivatives of equation (A6) with respect to θ vanish leads to other equations for F_n and, after $n + 1$ such differentiations, the equations acquire forms for which solutions are known. The value of F_n can then be determined from these known solutions by evaluating each of the derivatives $\dfrac{d^K w_n}{d\theta^K}$ at any arbitrary value of θ in the range $-C < \theta < C$ as follows:

The Kth derivative of w_n with respect to θ (equation (A6)) gives the following integral equation for F_n:

$$\frac{d^K w_n}{d\theta^K} = -\frac{\beta^2(n+2)}{\pi} \int_{-C}^{C} \frac{F_n(\sigma)\,d\sigma}{\left(1-\beta^2\sigma^2\right)^{3/2}} \int_0^{\cosh^{-1}\frac{q}{N_0}} \left(\frac{\beta x}{1-\beta^2\sigma^2}\right)^K \left(q - \right.$$

$$\left. N_0\cosh\tau\right)^{n+1} \frac{\cosh\tau}{N_0}\,\Phi(\tau)\,d\tau = 0 \tag{A7}$$

where

$$\Phi(\tau) = \frac{(n+1)!}{(n+1-K)!}\left(\frac{\beta\sigma + \cosh\tau}{N_0 \cosh\tau - q}\right)^K -$$

$$\frac{K(n+1)!}{(n+2-K)!}\left(\frac{\beta\sigma + \cosh\tau}{N_0 \cosh\tau - q}\right)^{K-1} N_0^{-1} +$$

$$\frac{K(K-1)(n+1)!}{(n+3-K)!}\left(\frac{\beta\sigma + \cosh\tau}{N_0 \cosh\tau - q}\right)^{K-2} N_0^{-2} + \ldots + (-1)^K K! N_0^{-K}$$

When $K = n + 1$, the expression for $\Phi(\tau)$ may be recognized as being a binomial expression, namely

$$\Phi(\tau) = (n+1)!\left(\frac{\beta\sigma + \cosh\tau}{N_0 \cosh\tau - q} - \frac{1}{N_0}\right)^{n+1}$$

$$= \frac{(n+1)!(\beta\sigma N_0 + q)^{n+1}}{(-N_0)^{n+1}(q - N_0 \cosh\tau)^{n+1}}$$

$$= \frac{(n+1)!x^{n+1}}{(-N_0)^{n+1}(q - N_0 \cosh\tau)^{n+1}} \tag{A8}$$

Thus the integral equation corresponding to the $n + 1$ derivative of w_n is

$$\frac{d^{n+1}w_n}{d\theta^{n+1}} = \frac{(-1)^{n+1}(n+2)!\,\beta^{n+3}x^{2n+2}}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)\,d\sigma}{N_0^{\,n+2}(1-\beta^2\sigma^2)^{n+\frac{5}{2}}}\int_0^{\sinh^{-1}\frac{x\sqrt{1-\beta^2\theta^2}}{N_0\sqrt{1-\beta^2\sigma^2}}}\cosh\tau\,d\tau$$

$$= \frac{(-1)^{n+1}(n+2)!\,x^n\sqrt{1-\beta^2\theta^2}}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)}{(\theta-\sigma)^{n+3}}\,d\sigma = 0 \qquad (A9)$$

Further differentiation of w_n leads to other equations involving integrals similar to that in equation (A9) and, as will be seen subsequently, is not necessary for the determination of F_n. The singularity at $\sigma = \theta$ in equation (A9) is a result of going to the limit $z = 0$ after equation (A5) and, as previously implied, is to be ignored.

Consider the following equation similar to equation (A9):

$$\int_{-C}^{C}\frac{f_m(\sigma)\,d\sigma}{(\theta-\sigma)^{n+3}} = 0 \qquad (A10)$$

It is known by analogy with problems in incompressible flow and may be shown by direct substitution and reduction that this equation is satisfied for any value of n $(n = 0, 1, 2, \ldots)$ by the function

$$f_0(\sigma) = \sqrt{C^2 - \sigma^2} \qquad (A11)$$

This function also satisfies equation (A9) for any value of n and satisfies equation (A7) for all derivatives $\dfrac{d^K w_n}{d\theta^K}$ beyond the nth derivative. A more general form of solution to equation (A10) may be shown to be

$$f_m = \sigma^m \sqrt{c^2 - \sigma^2} \qquad (A12)$$

where m is an integer and $m \lesseqgtr n$. The validity of this solution follows, since

$$\sigma^m = \left[\theta - (\theta - \sigma)\right]^m = \sum_{r=0}^{m} \frac{m!\,\theta^{m-r}}{(m - r)!\,r!} (\theta - \sigma)^r \qquad (A13)$$

and

$$\frac{\sigma^m \sqrt{c^2 - \sigma^2}}{(\theta - \sigma)^{n+3}} = \sum_{r=0}^{m} \frac{m!\,\theta^{m-r}}{(m - r)!\,r!} \frac{\sqrt{c^2 - \sigma^2}}{(\theta - \sigma)^{n-r+3}} \qquad (A14)$$

Each term of the summation in equation (A14) is found to have the form of the integrand in equation (A10), from which the function (A12) is concluded to satisfy equation (A10) and equation (A7) for all values of $K > n$.

From the foregoing discussion and consideration the distribution function $F_n(\sigma)$ may be uniquely determined in terms of expression (A12) as follows:

Consider the expression

$$F_n(\sigma) = A_0 f_0 + A_1 f_1 + \ldots + A_n f_n \qquad (A15)$$

where the coefficients A_0, A_1, . . . A_n are constants that are to be determined.

Each term on the right of equation (A15) is noted to satisfy equation (A7) for all derivatives beyond the nth and a total of

n + 1 parameters is to be determined. If expression (A15) is intro-
duced into equations (A6) and (A7) and the indicated integrations in
equation (A6) and in equation (A7) for $K = 1, 2, \ldots n$ are performed,
n + 1 linear equations are obtained in $A_0, A_1, \ldots A_n$ from which
these constants may be determined.

The integrations with respect to σ in equations (A6) and (A7) are
in general difficult and tedious to perform; however, as previously
pointed out it is only necessary to perform the integrations for some
particular value of θ in the range $-C < \theta < C$. The integrals have
their simplest form when $\theta = 0$ and the integrations can be made for
this value of θ by reductions and use of formulas in reference 11.

The distribution functions $F_0(\sigma)$, $F_1(\sigma)$, $F_2(\sigma)$, $F_3(\sigma)$, $\overline{F}_0(\sigma)$,
and $\overline{F}_1(\sigma)$ and functions $D_n(\xi, \eta)$ calculated by the foregoing method are
as follows:

$$F_0 = A_0\sqrt{C^2 - \sigma^2} \qquad\qquad D_0 = A_0\sqrt{C^2\xi^2 - \eta^2}$$

$$F_1 = A_1\sqrt{C^2 - \sigma^2} \qquad\qquad D_1 = A_1\sqrt{C^2\xi^2 - \eta^2}$$

$$F_2 = (A_2 + A_3\beta^2\sigma^2)\sqrt{C^2 - \sigma^2} \qquad\qquad D_2 = \left(A_2 + A_3\frac{\beta^2\eta^2}{\xi^2}\right)\sqrt{C^2\xi^2 - \eta^2}$$

$$F_3 = (A_4 + A_5\beta^2\sigma^2)\sqrt{C^2 - \sigma^2} \qquad\qquad D_3 = \left(A_4 + \frac{A_5\beta^2\eta^2}{\xi^2}\right)\sqrt{C^2\xi^2 - \eta^2}$$

$$\overline{F}_0 = (A_6 + A_7\beta^2\sigma^2)\sqrt{C^2 - \sigma^2} \qquad\qquad \overline{D}_0 = \left(A_6 + A_7\frac{\beta^2\eta^2}{\xi^2}\right)\sqrt{C^2\xi^2 - \eta^2}$$

$$\overline{F}_1 = (A_8 + A_9\beta^2\sigma^2)\sqrt{C^2 - \sigma^2} \qquad\qquad \overline{D}_1 = \left(A_8 + A_2\frac{\beta^2\eta^2}{\xi^2}\right)\sqrt{C^2\xi^2 - \eta^2}$$

where

$$A_0 = \frac{1}{E'}$$

$$A_1 = \frac{1 - \beta^2 C^2}{\beta^2 C^2 F' + (1 - \beta^2 C^2)E'}$$

$$A_2 = \frac{2d_3}{2d_1d_3 + \beta^2c^2d_2{}^2}$$

$$A_3 = \frac{2d_2}{2d_1d_3 + \beta^2c^2d_2{}^2}$$

$$A_4 = \frac{6d_6}{6d_4d_6 + \beta^2c^2d_5{}^2}$$

$$A_5 = \frac{6d_5}{6d_4d_6 + \beta^2c^2d_5{}^2}$$

$$A_6 = \frac{2d_3d_7 - \beta^2c^2d_2d_8}{2d_1d_3 + \beta^2c^2d_2{}^2}$$

$$A_7 = \frac{2(d_1d_8 + d_2d_7)}{2d_1d_3 + \beta^2c^2d_2{}^2}$$

$$A_8 = \frac{6d_6d_9 - \beta^2c^2d_2d_5}{6d_4d_6 + \beta^2c^2d_5{}^2}$$

$$A_9 = \frac{6(d_2d_4 + d_5d_9)}{6d_4d_6 + \beta^2c^2d_5{}^2}$$

The quantities d_n are defined as

$$d_1 = \frac{5(\beta^2c^2 - 3\beta^4c^4)F' + (2 - 10\beta^2c^2 + 6\beta^4c^4)E'}{2(1 - \beta^2c^2)^2}$$

$$d_2 = \frac{2\beta^2 c^2 F' - (1 + \beta^2 c^2)E'}{(1 - \beta^2 c^2)^2}$$

$$d_3 = \frac{(3\beta^2 c^2 - 5\beta^4 c^4)F' - (6 - 10\beta^2 c^2 + 2\beta^4 c^4)E'}{(1 - \beta^2 c^2)^2}$$

$$d_4 = \frac{(27\beta^2 c^2 - 31\beta^4 c^4 + 12\beta^6 c^6)F' + (6 - 55\beta^2 c^2 + 65\beta^4 c^4 - 24\beta^6 c^6)E'}{6(1 - \beta^2 c^2)^3}$$

$$d_5 = \frac{(9\beta^2 c^2 - \beta^4 c^4)F' - (3 + 7\beta^2 c^2 - 2\beta^4 c^4)E'}{(1 - \beta^2 c^2)^3}$$

$$d_6 = \frac{(3\beta^2 c^2 - 9\beta^4 c^4 - 2\beta^6 c^6)F' - (6 - 15\beta^2 c^2 + 5\beta^4 c^4 - 4\beta^6 c^6)E'}{(1 - \beta^2 c^2)^3}$$

$$d_7 = \frac{\beta^2 c^2(E' - F')}{2(1 - \beta^2 c^2)}$$

$$d_8 = \frac{\beta^2 c^2 F' - E'}{1 - \beta^2 c^2}$$

$$d_9 = \frac{\beta^2 c^2 \left[(\beta^2 c^2 - 3)F' + (4 - 2\beta^2 c^2)E'\right]}{6(1 - \beta^2 c^2)^2}$$

Combinations of these functions useful for writing the potential in concise form are:

$$\frac{A_0 - A_1}{2} = \lambda_1$$

$$\frac{A_1 - A_2}{2} = \lambda_2$$

$$\frac{A_2 - A_4}{2} = \lambda_3$$

$$\frac{2A_3 - A_5}{2} = \lambda_4$$

$$\lambda_1 - \lambda_2 = \lambda_5$$

$$\lambda_2 - \lambda_3 = \lambda_6$$

$$\lambda_5 - \lambda_6 = \lambda_7$$

$$\lambda_4 + \frac{A_3}{2} = \lambda_8$$

$$\frac{A_0 A_6}{2} = \lambda_9$$

$$\frac{A_0 A_7}{2} = \lambda_{10}$$

$$\frac{A_1 A_8}{2} = \lambda_{11}$$

$$\frac{A_1 A_9}{2} = \lambda_{12}$$

$$\lambda_9 - \lambda_{11} = \lambda_{13}$$

$$\lambda_{10} - \lambda_{12} = \lambda_{14}$$

$$\lambda_5 - \frac{\lambda_9}{M^2} = \sigma_1$$

$$\frac{A_3}{2} - \frac{\lambda_{10}}{M^2} = \sigma_2$$

$$\frac{\lambda_7}{3} - \frac{\lambda_{13}}{M^2} = \sigma_3$$

$$\frac{\lambda_8}{2} - \frac{\lambda_{14}}{M^2} = \sigma_4$$

$$\lambda_6 - \frac{\lambda_{11}}{M^2} = \sigma_5$$

$$\lambda_4 + \frac{\lambda_{12}}{M^2} = \sigma_6$$

It will be noted that the quantities A_i ($i = 0, 1, \ldots 9$) and λ_j ($j = 1, 2, \ldots 14$) are functions only of the product βC. They are shown plotted in figures 3 and 4, respectively. The quantities σ_k ($k = 1, 2, \ldots 6$) are functions of Mach number M and the product βC and can be evaluated for particular values of M from the plots of A_i and λ_j.

REFERENCES

1. Garrick, I. E., and Rubinow, S. I.: Theoretical Study of Air Forces
   on an Oscillating or Steady Thin Wing in a Supersonic Main Stream.
   NACA Rep. 872, 1947. (Formerly NACA TN 1383.)

2. Watkins, Charles E.: Effect of Aspect Ratio on Undamped Torsional
   Oscillations of a Thin Rectangular Wing in Supersonic Flow. NACA
   TN 1895, 1949.

3. Watkins, Charles E.: Effect of Aspect Ratio on the Air Forces and
   Moments of Harmonically Oscillating Thin Rectangular Wings in
   Supersonic Potential Flow. NACA TN 2064, 1950.

4. Robinson, A.: On Some Problems of Unsteady Supersonic Aerofoil Theory.
   Rep. No. 16, College of Aero., Cranfield (British), May 1948.

5. Haskind, M. D., and Falkovich, S. V.: Vibration of a Wing of Finite
   Span in a Supersonic Flow. NACA TM 1257, 1950.

6. Stewartson, K.: On Linearized Potential Theory of Unsteady Supersonic
   Motion. Quarterly Jour. of Mech. and Applied Math. (Clarendon
   Press), vol. III, pt. 2, June 1950, pp. 182-199.

7. Robinson, A.: On Source and Vortex Distributions in the Linearized
   Theory of Steady Supersonic Flow. Rep. No. 9, College of Aero.,
   Cranfield (British), Oct. 1947.

8. Brown, Clinton E.: Theoretical Lift and Drag of Thin Triangular Wings
   at Supersonic Speeds. NACA Rep. 839, 1946. (Formerly NACA TN 1183.)

9. Brown, Clinton E., and Adams, Mac C.: Damping in Pitch and Roll of
   Triangular Wings at Supersonic Speeds. NACA Rep. 892, 1948.
   (Formerly NACA TN 1566.)

10. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at
    Speeds below and above the Speed of Sound. NACA Rep. 835, 1946.
    (Formerly NACA TN 1032.)

11. De Haan, D. Bierens: Nouvelles Tables D'Intégrales Définies.
    G. E. Stechert & Co. (New York), 1939, pp. 474-476.
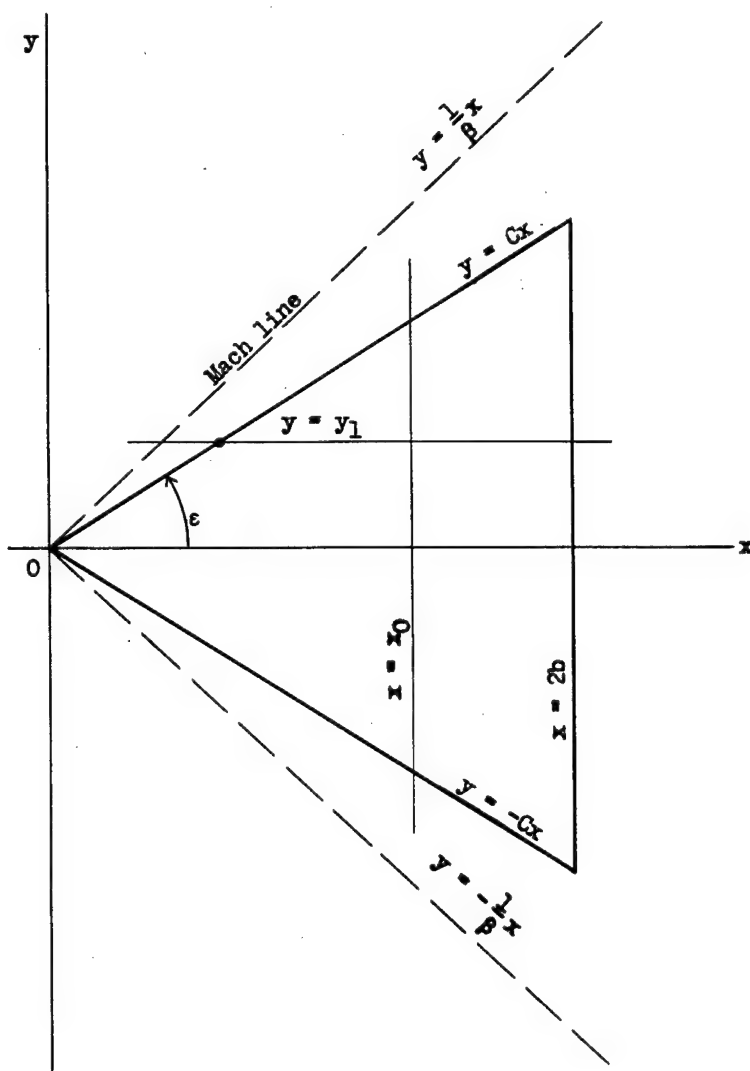
(a) Plan form (xy-plane).
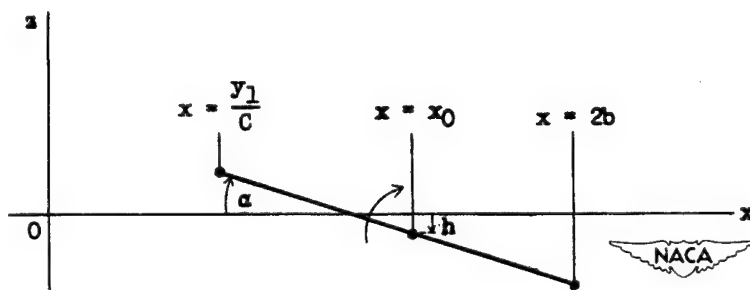
(b) Section $y = y_1$ (xz-plane).

Figure 1.- Sketch illustrating coordinate system and the two degrees of
freedom α and h.
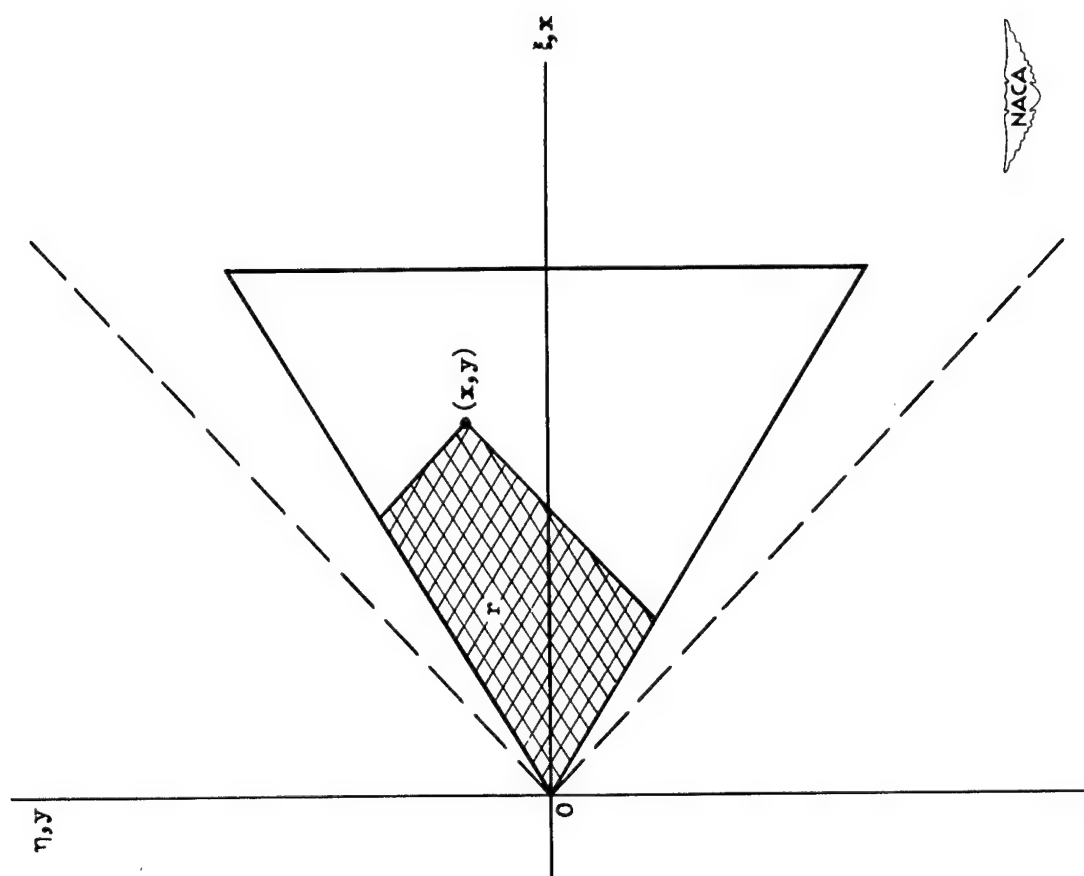
Figure 2.- Sketch illustrating region of integration for the velocity
potential.

Figure 3.- Variation of the quantities A_i as functions of βC.
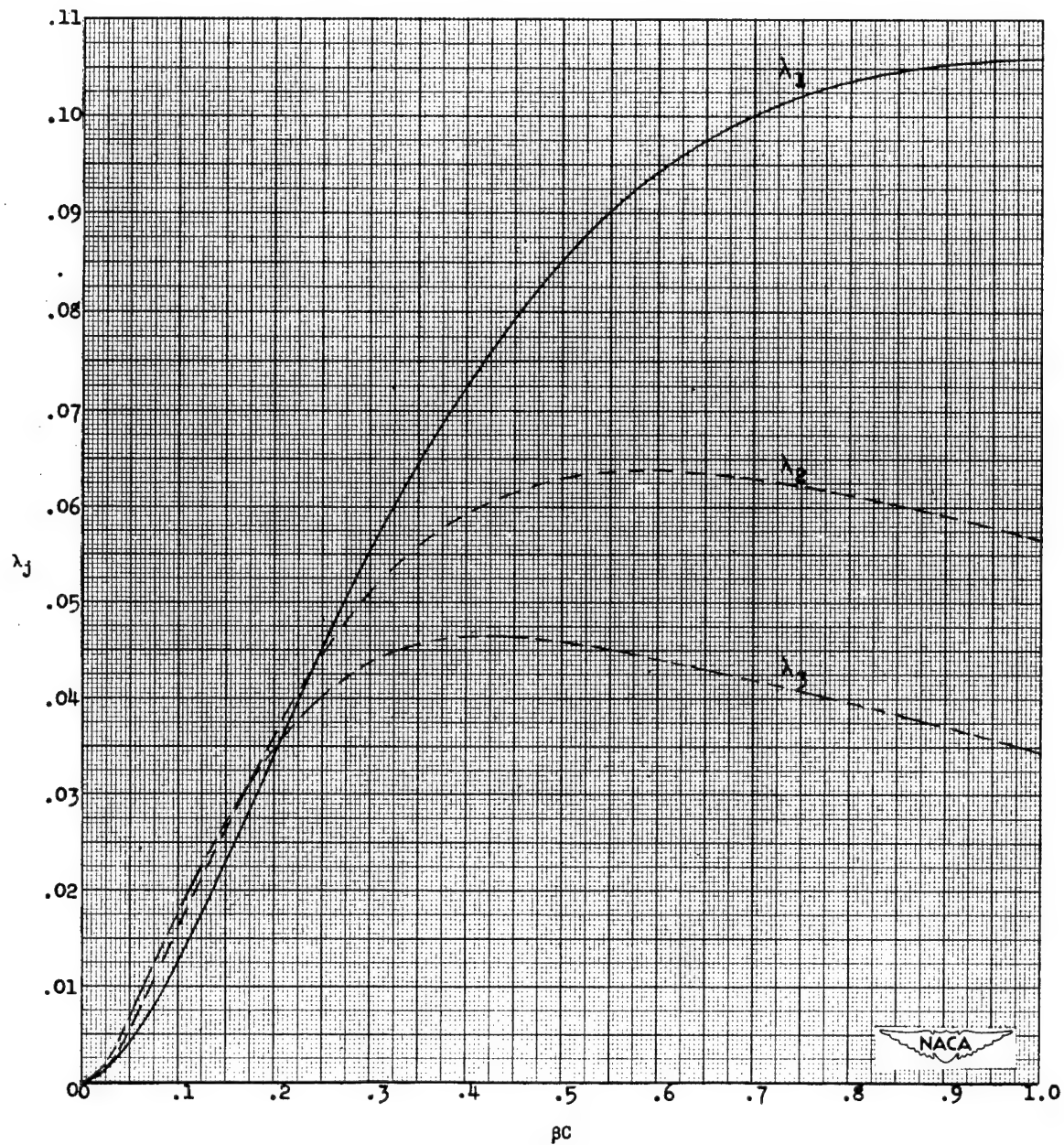
(a) j = 1, 2, and 3.

Figure 4.- Variation of the quantities λ_j as functions of βC.
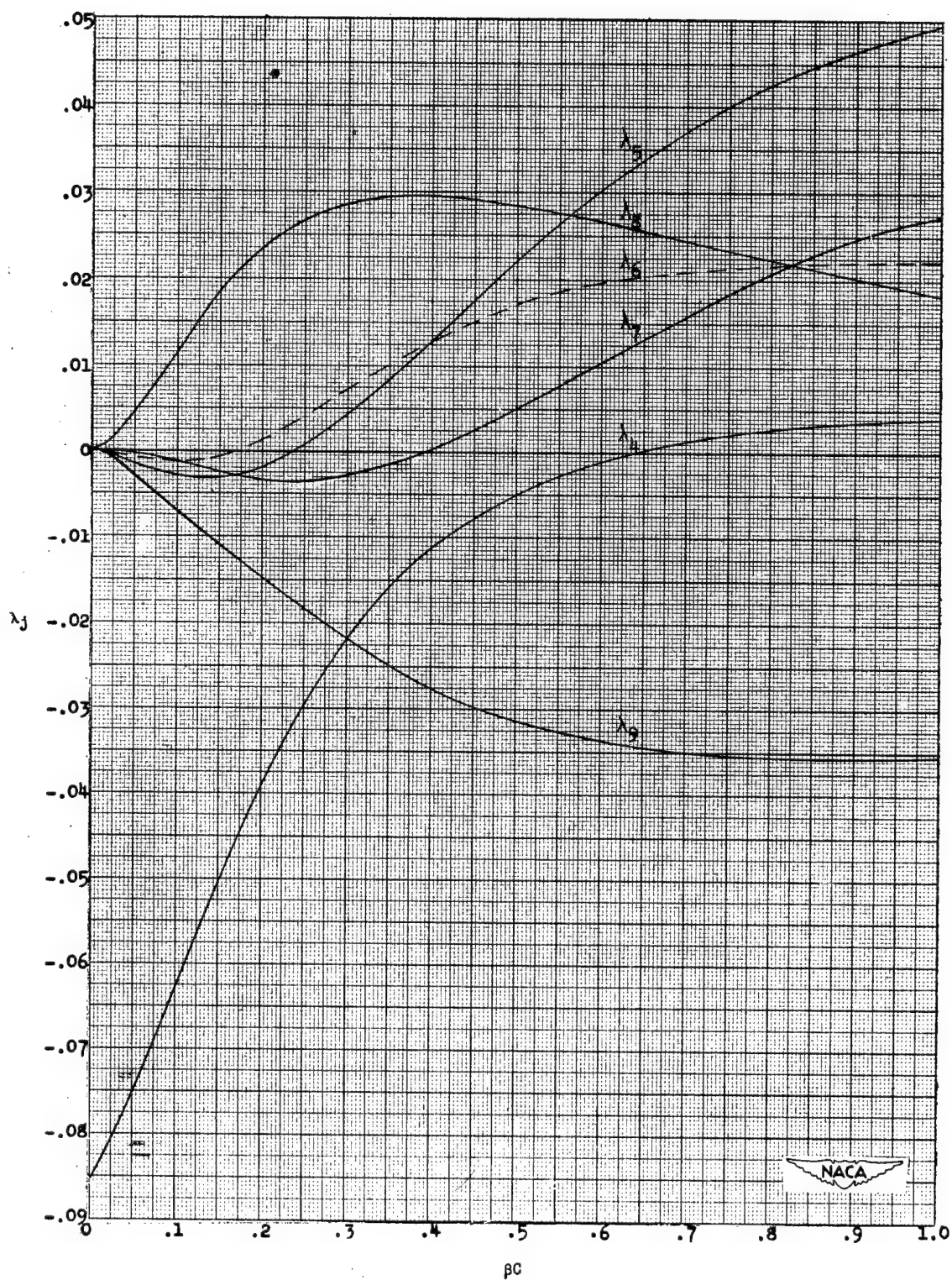
(b)   j = 4, 5, 6, 7, 8, and 9.

Figure 4.- Continued.
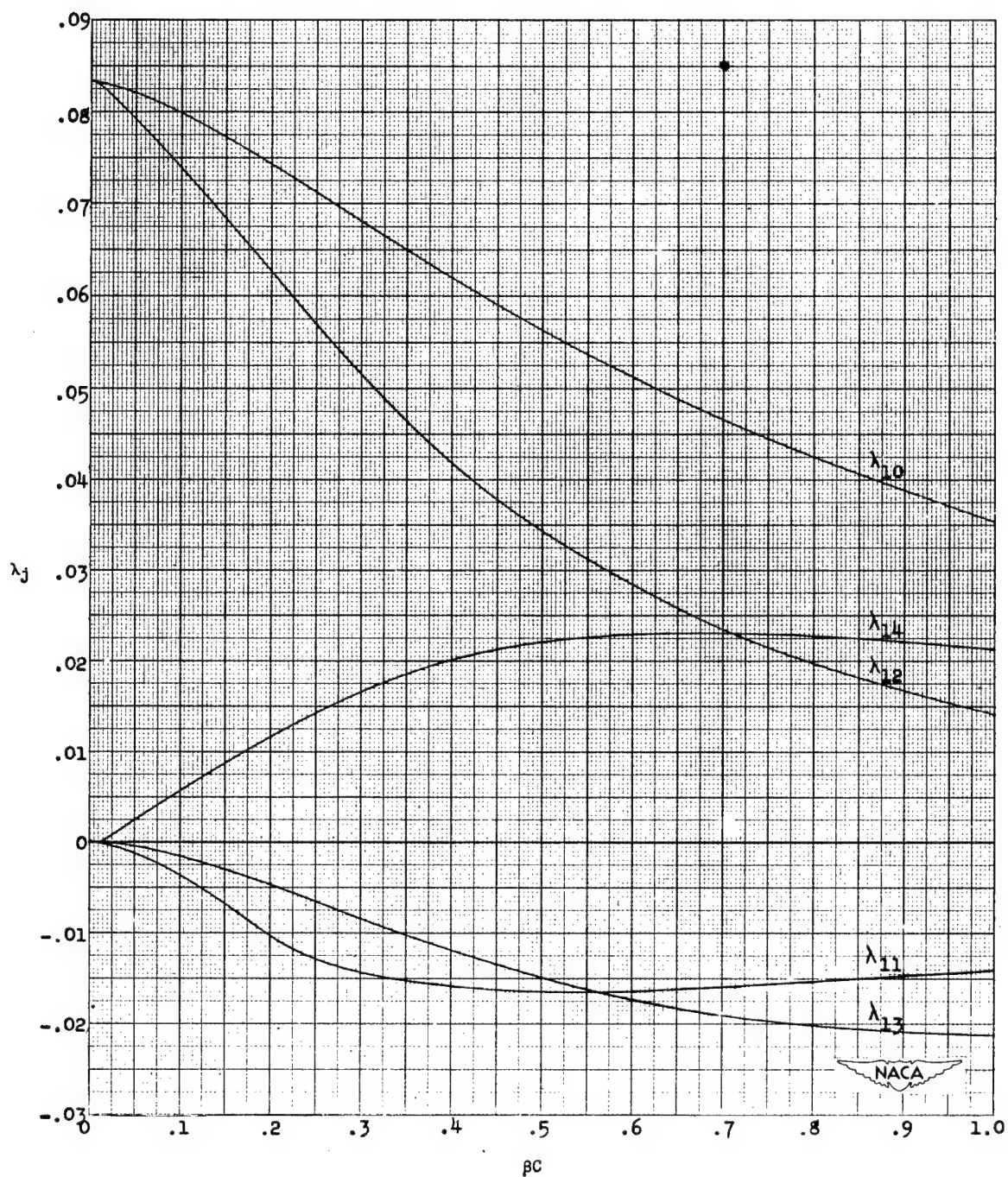
(c)   j = 10, 11, 12, 13, and 14.

Figure 4.- Concluded.
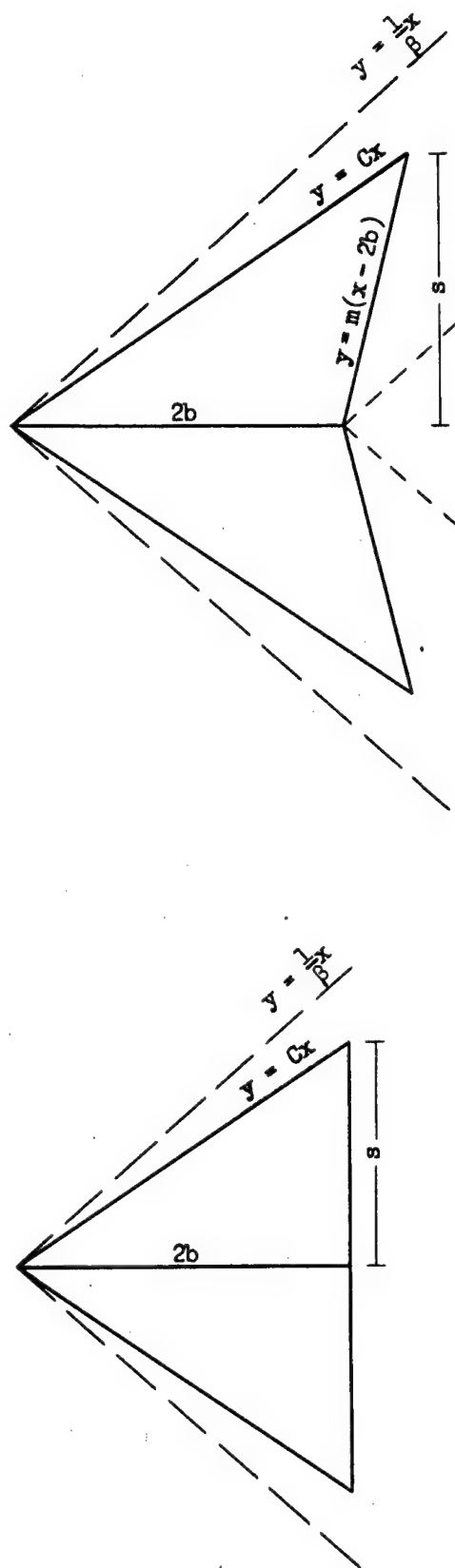
(a) Plan form A.
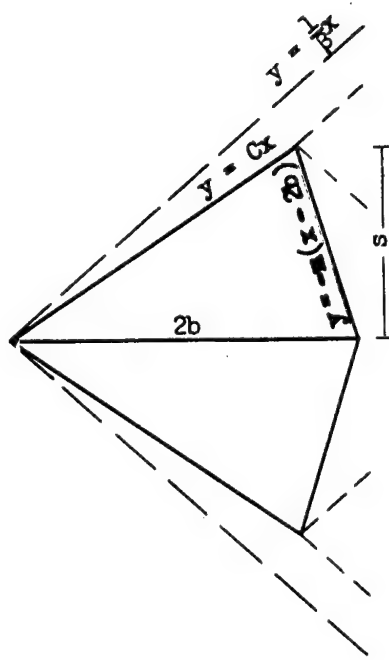
(b) Plan form B.
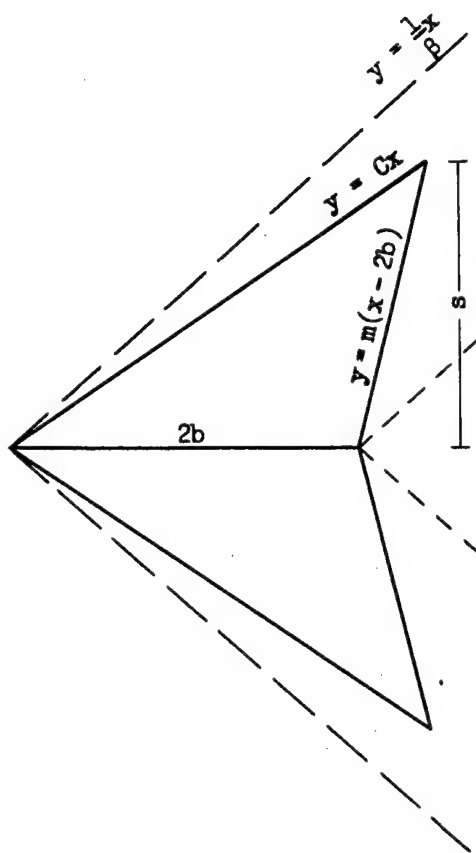
(c) Plan form C.
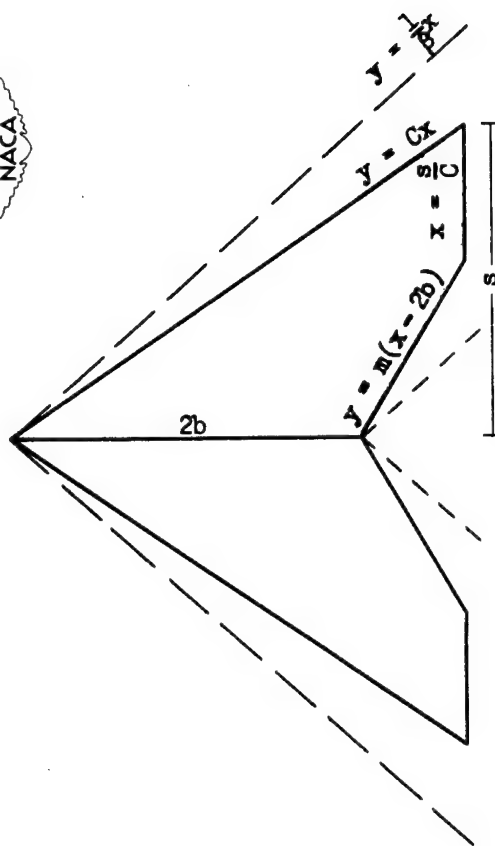
(d) Plan form D.

Figure 5.- Sketches illustrating different plan forms for which the force equations (38) and moment equations (40) apply.
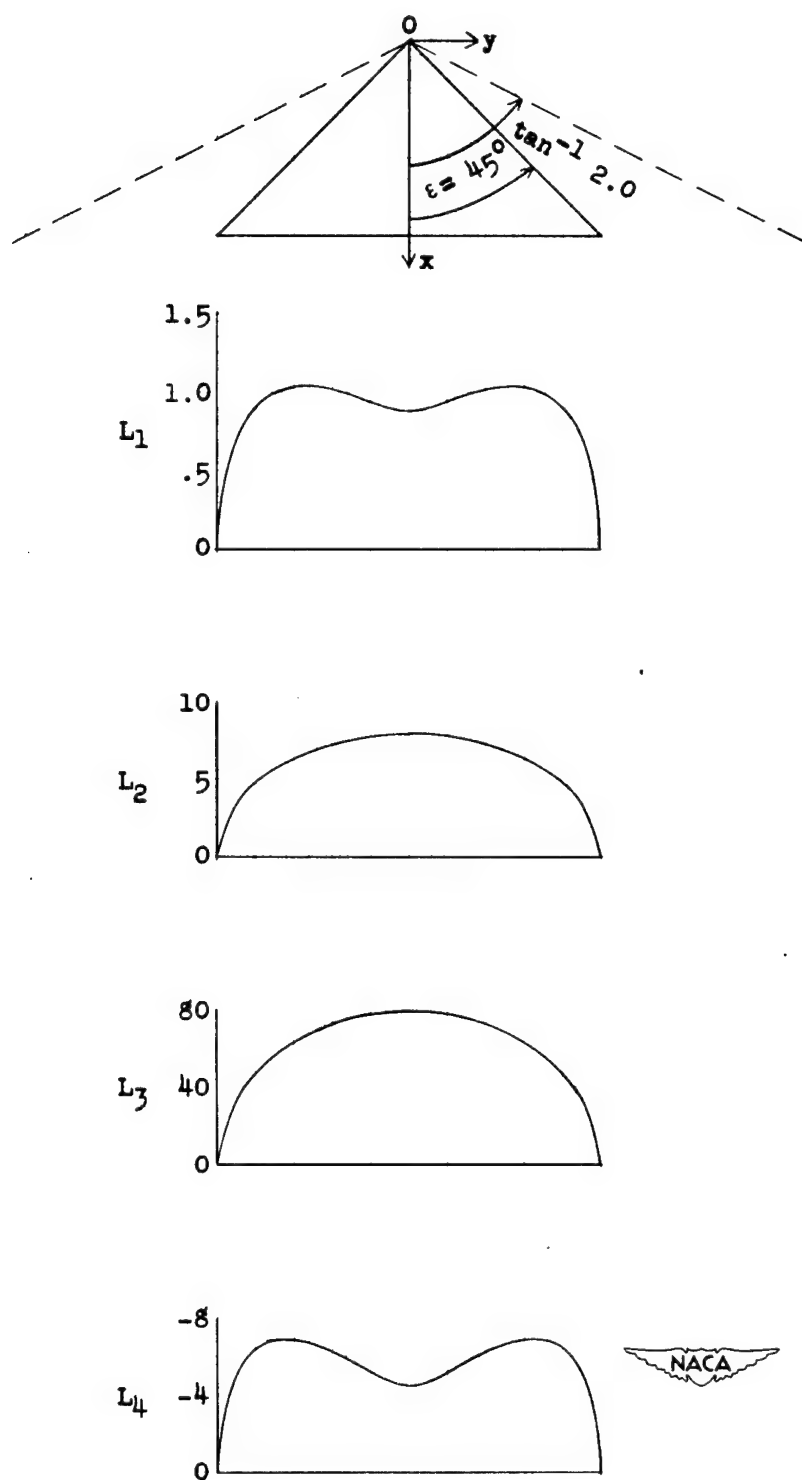
(a) Lift force.

Figure 6.- Spanwise distribution of components of lift force and moment
   coefficients for $x_0 = 0.6$. $\beta C = 0.5$; $C = 1.0$; $k = 0.1$; $M = \sqrt{5/4}$.
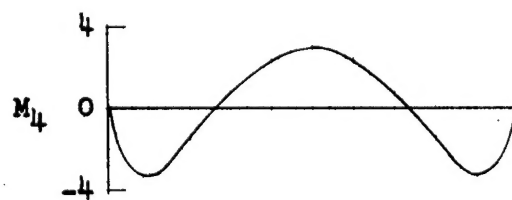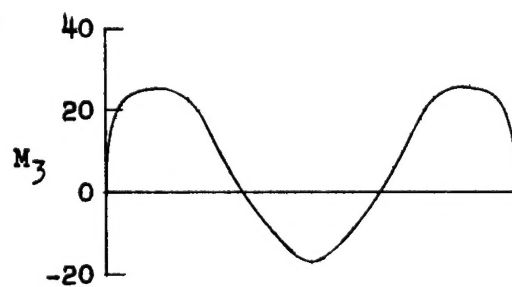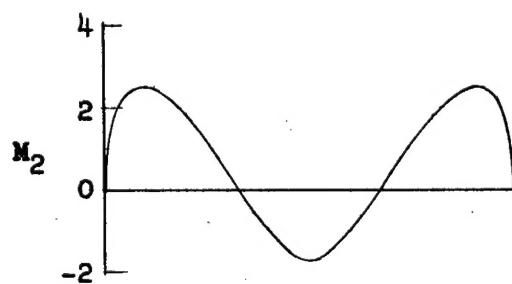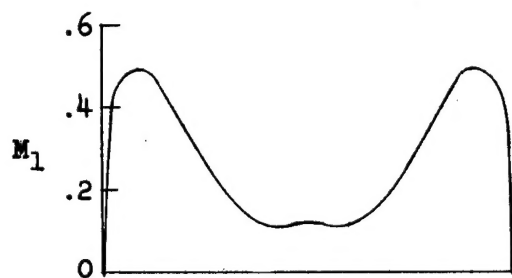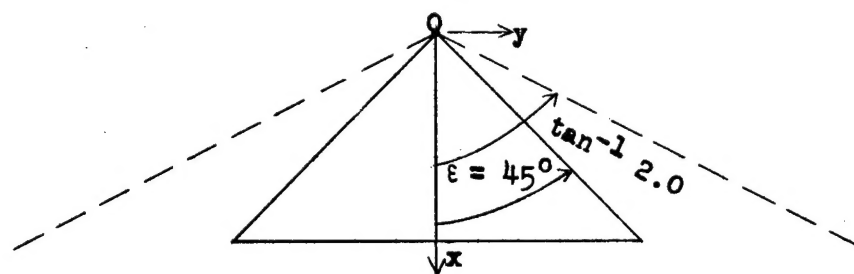
(b) Moment.

Figure 6.- Concluded.
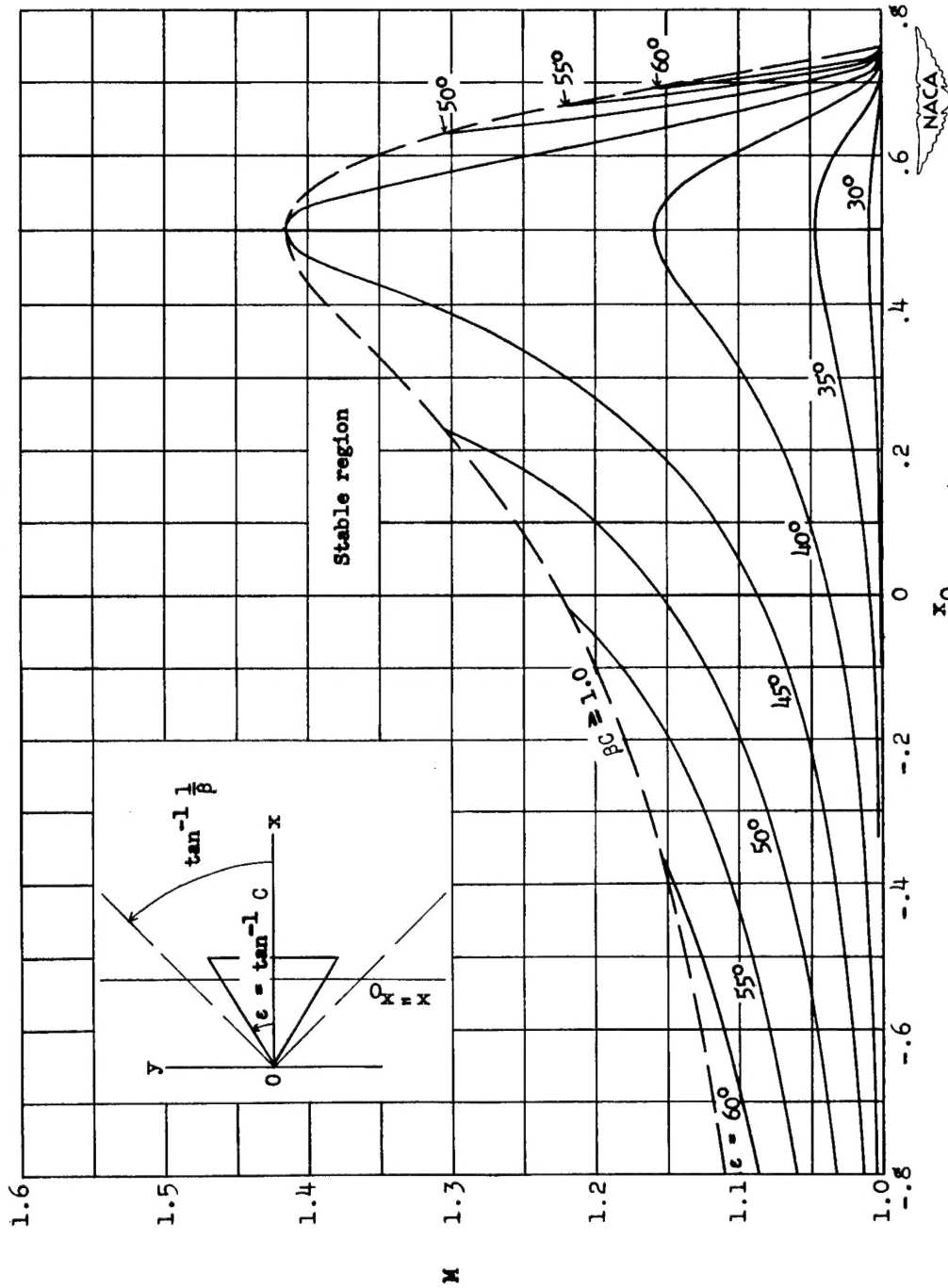
Figure 7.- Curves showing ranges of Mach number M and axis of rotation x_0 for which the aerodynamic torsional damping moment vanishes for some selected values of angle ϵ.

NACA TN 2457

which is sufficient for a large class of practical ap-
plications, are given for the velocity potential and
components of chordwise section force and moment
coefficients.

Copies obtainable from NACA, Washington

NACA TN 2457

which is sufficient for a large class of practical ap-
plications, are given for the velocity potential and
components of chordwise section force and moment
coefficients.

Copies obtainable from NACA, Washington

NACA TN 2457

which is sufficient for a large class of practical ap-
plications, are given for the velocity potential and
components of chordwise section force and moment
coefficients.

Copies obtainable from NACA, Washington

NACA TN 2457

which is sufficient for a large class of practical ap-
plications, are given for the velocity potential and
components of chordwise section force and moment
coefficients.

Copies obtainable from NACA, Washington